

# On the global solution problem for semilinear generalized Tricomi equations, I

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## Abstract

In this paper, we are concerned with the global Cauchy problem for the semilinear generalized Tricomi equation  $\partial_t^2 u - t^m \Delta u = |u|^p$  with initial data  $(u(0, \cdot), \partial_t u(0, \cdot)) = (u_0, u_1)$ , where  $t \geq 0$ ,  $x \in \mathbb{R}^n$  ( $n \geq 3$ ),  $m \in \mathbb{N}$ ,  $p > 1$ , and  $u_i \in C_0^\infty(\mathbb{R}^n)$  ( $i = 0, 1$ ). We show that there exists a critical exponent  $p_{\text{crit}}(m, n) > 1$  such that the solution  $u$ , in general, blows up in finite time when  $1 < p < p_{\text{crit}}(m, n)$ . We further show that there exists a conformal exponent  $p_{\text{conf}}(m, n) > p_{\text{crit}}(m, n)$  such that the solution  $u$  exists globally when  $p > p_{\text{conf}}(m, n)$  provided that the initial data is small enough. In case  $p_{\text{crit}}(m, n) < p \leq p_{\text{conf}}(m, n)$ , we will establish global existence of small data solutions  $u$  in a subsequent paper [13].

**Keywords.** Generalized Tricomi equation, critical exponent, conformal exponent, global existence, blowup, Strichartz estimate.

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## 1 Introduction

In this paper, we are concerned with the global existence and blowup of solutions  $u$  of the semilinear generalized Tricomi equation

$$\begin{cases} \partial_t^2 u - t^m \Delta u = |u|^p, & (t, x) \in \mathbb{R}_+^{n+1}, \\ u(0, \cdot) = u_0(x), \quad \partial_t u(0, \cdot) = u_1(x). \end{cases} \quad (1.1)$$

Here,  $t \geq 0$ ,  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$  ( $n \geq 3$ ),  $m \in \mathbb{N}$ ,  $p > 1$ , and  $u_i \in C_0^\infty(B(0, M))$  ( $i = 0, 1$ ), where  $B(0, M) = \{x: |x| < M\}$ , and  $M > 0$ . In general, one has only weak solutions of (1.1) since the nonlinear term  $|u|^p$  is not  $C^2$  when  $1 < p < 2$ . For the local existence and regularity of solutions  $u$

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of (1.1) under weaker regularity assumptions on  $(u_0, u_1)$ , the reader may consult [21–24, 30] and the references given therein; here we shall not discuss this problem.

Our present objective is, for given  $m \in \mathbb{N}$  and  $n \geq 3$ , to determine a critical exponent  $p_{\text{crit}}(m, n) > 1$  such that solutions  $u$  of (1.1) will, in general, blow up in finite time when  $1 < p < p_{\text{crit}}(m, n)$  and a conformal exponent  $p_{\text{conf}}(m, n) > p_{\text{crit}}(m, n)$  with the property that small data solutions  $u$  of (1.1) exist globally in time when  $p > p_{\text{conf}}(m, n)$ . Global existence of small data solutions  $u$  of (1.1) for  $p$  in the range  $p_{\text{crit}}(m, n) < p \leq p_{\text{conf}}(m, n)$  will be established in a forthcoming paper [13].

Before we describe the content of this paper in detail, we recall a number of related results. Firstly, we consider the semilinear wave equation

$$\begin{cases} \partial_t^2 u - \Delta u = |u|^p, & (t, x) \in \mathbb{R}_+^{n+1}, \\ u(0, \cdot) = u_0(x), \quad \partial_t u(0, \cdot) = u_1(x), \end{cases} \quad (1.2)$$

where  $p > 1$ ,  $n \geq 2$ , and  $u_i \in C_0^\infty(\mathbb{R}^n)$  ( $i = 0, 1$ ). Let  $p_1(n)$  denote the positive root of the quadratic equation

$$(n-1)p^2 - (n+1)p - 2 = 0. \quad (1.3)$$

Strauss [28] made the following conjecture:

**Strauss Conjecture.** If  $p > p_1(n)$ , then small data solutions of problem (1.2) exist globally. If  $1 < p < p_1(n)$ , then small data solutions of problem (1.2) blow up in finite time.

For  $1 < p \leq p_1(n)$  and non-negative initial data  $(u_0, u_1)$ , blowup for the solution  $u$  of (1.2) has been established, while, for  $p > p_1(n)$ , global existence of small data solution  $u$  of (1.2) has also been systematically studied (see [9, 11, 12, 15, 18, 25, 26, 31, 32] and the references therein). Especially, in [9] and [18], one finds a detailed history of results related to the Strauss Conjecture.

Secondly, we consider the semilinear wave equation with time-dependent dissipation

$$\begin{cases} \partial_t^2 u - \Delta u + \frac{\mu}{(1+t)^\alpha} \partial_t u = |u|^p, & (t, x) \in \mathbb{R}_+^{n+1}, \\ u(0, \cdot) = u_0(x), \quad \partial_t u(0, \cdot) = u_1(x), \end{cases} \quad (1.4)$$

where  $\mu > 0$ ,  $\alpha \geq 0$ ,  $p > 1$ ,  $n \geq 1$ , and  $u_i \in C_0^\infty(\mathbb{R}^n)$  ( $i = 0, 1$ ). Define the Fujita exponent  $p_2(n) = 1 + \frac{2}{n}$  as in [8]. It follows from well-known results that for the semi-linear heat equation  $\partial_t u - \Delta u = |u|^p$  with initial data  $u(0, \cdot) = u_0(x)$ , small data solution  $u$  exists globally if  $u_0$  is sufficiently small and  $p > p_2(n)$ ; otherwise, solutions  $u$  will, in general, blow up in finite time.

As for problem (1.4), the following result has been established in a series of papers [4, 5, 17, 19, 20]:

**Theorem A.** (i) For  $0 \leq \alpha < 1$ , if  $p > p_2(n)$ , then (1.4) has a global small data solution; if  $1 < p \leq p_2(n)$ , the solution  $u$  of (1.4) generally blows up in finite time.

(ii) For  $\alpha > 1$ , or  $\alpha = 1$  and  $0 < \mu \ll 1$ , then the properties of problem (1.4) are analogous to those of problem (1.2).

(iii) For  $\alpha = 1$  and  $\mu \gg 1$ , then the properties of problem (1.4) are analogous to those of the semi-linear heat equation  $\partial_t u - \Delta u = |u|^p$ .

*Remark 1.1.* Note that for  $\alpha = 1$  and  $\mu \approx 1$ , it is still an interesting open problem to determine explicitly a critical value  $p_c(n)$  so that problem (1.4) has global small data solutions for  $p > p_c(n)$ , while solutions of (1.4), in general, blow up in finite time when  $1 < p \leq p_c(n)$ .

Thirdly, we consider the semilinear generalized Tricomi equation

$$\begin{cases} \partial_t^2 u - t^{2k} \Delta u = |u|^p, & (t, x) \in \mathbb{R}_+^{n+1}, \\ u(0, \cdot) = u_0(x), \quad \partial_t u(0, \cdot) = u_1(x), \end{cases} \quad (1.5)$$

where  $k > \frac{1}{2}$  is a real constant,  $p > 1$ ,  $n \geq 1$ , and  $u_i \in C_0^\infty(\mathbb{R}^n)$  ( $i = 0, 1$ ). Note that problems (1.4) and (1.5) are closely related for large  $t > 0$ . Indeed, with  $T = t^{k+1}/(k+1)$ , the equation in (1.5) becomes

$$\partial_T^2 u - \Delta u + \frac{k}{k+1} \frac{\partial_T u}{T} = (k+1)^{-\frac{2k}{k+1}} T^{-\frac{2k}{k+1}} |u|^p,$$

which is essentially the equation

$$\partial_t^2 u - \Delta u + \frac{\mu_k}{1+t} \partial_t u = C_k (1+t)^{-\frac{2k}{k+1}} |u|^p \quad (1.6)$$

for large  $t > 0$  with  $\mu_k = \frac{k}{k+1}$  and  $C_k = (k+1)^{-\frac{2k}{k+1}}$ .

Comparing the equations in (1.4) and (1.6), one realizes that their linear parts are identical. Note that the coefficient  $\mu_k$  in (1.6) can be arbitrarily close to 1 when  $k$  is large. In this case, however, it is unknown what the critical value of the exponent  $p$  for problem (1.4) is. This especially means that the methods of [4, 5, 17, 19, 20] are not applicable for studying problem (1.5).

We now recall some known results concerning problem (1.5). Under the conditions

$$\begin{cases} \frac{(n+1)(p-1)}{p+1} \leq \frac{k}{k+1}, \\ \left( \frac{2}{p-1} - \frac{n(k+1)}{p+1} \right) p \leq 1, \\ \frac{p+1}{p(p-1)n(k+1)} \leq \frac{1}{p+1} \leq \frac{k+2}{(n+1)(p-1)(k+1)} \end{cases} \quad (1.7)$$

(corresponding to (1.8) and (1.12) of [30] with  $\alpha = p-1$  and  $\beta = \frac{2}{p-1} - \frac{n(k+1)}{p+1}$ ) it was shown in [30, Theorem 1.2] that problem (1.5) has a global small data solution  $u \in C([0, \infty), L^{p+1}(\mathbb{R}^n)) \cap C^1([0, \infty), \mathcal{D}'(\mathbb{R}^n))$ . On the other hand, under the conditions  $\int_{\mathbb{R}^n} u_1(x) dx > 0$  and

$$1 < p < \frac{(k+1)n+1}{(k+1)n-1}, \quad (1.8)$$

it was shown in [30, Theorem 1.3] that problem (1.5) has no global solution  $u \in C([0, \infty), L^{p+1}(\mathbb{R}^n))$ . Here we point out that (1.8) comes from condition (1.15) of [30]. In particular, for  $n = 3$ , from (1.7) and (1.8) one has (see also (1.16) of [30]):

**Theorem B.** *Let  $n = 3$ .*

- (i) *If  $\frac{3k+5+\sqrt{9k^2+42k+33}}{6k+4} < p < \min \left\{ \frac{3k+5}{3k+1}, \frac{5k+4}{3k+4} \right\}$ , then problem (1.5) admits a global small data solution  $u \in C([0, \infty), L^{p+1}(\mathbb{R}^3))$ .*
- (ii) *If  $1 < p < \frac{3k+4}{3k+2}$ , then, in general, the solution of problem (1.5) will blow up in finite time.*

Based on this theorem, Yagdjian [30] put forward the following conjecture (which corresponds to (1.17) of [30] with  $\alpha = p-1$ ):

**Conjecture.** Let  $n = 3$  and  $\frac{3k+4}{3k+2} \leq p \leq \frac{3k+5+\sqrt{9k^2+42k+33}}{6k+4}$ . Then small data solutions of problem (1.5) exists globally.

In this and a forthcoming paper [13], we will systematically study problem (1.1). In particular, our analysis will show that Yagdjian's conjecture fails in a certain range of  $p$ .

Let  $p_{\text{crit}}(m, n)$  be the positive root  $p$  of the quadratic equation

$$\left((m+2)\frac{n}{2} - 1\right)p^2 + \left((m+2)\left(1 - \frac{n}{2}\right) - 3\right)p - (m+2) = 0. \quad (1.9)$$

Note that  $p_{\text{crit}}(0, n) = p_1(n)$ , see (1.3). Then our first result asserts:

**Theorem 1.1** (Blow up for  $1 < p < p_{\text{crit}}(m, n)$ ). *Let  $1 < p < p_{\text{crit}}(m, n)$  and suppose that  $u_i \geq 0$  and  $u_i \not\equiv 0$  for  $i = 0, 1$ . Then problem (1.1) admits no global solution  $u$  with  $u \in C([0, \infty), H^1(\mathbb{R}^n)) \cap C^1([0, \infty), L^2(\mathbb{R}^n))$ .*

**Remark 1.2.** We have  $p_{\text{crit}}(2k, 3) = \frac{k+4+\sqrt{25k^2+48k+32}}{6k+4}$ , and it follows from a direct computation that

$$\frac{3k+4}{3k+2} < p_{\text{crit}}(2k, 3) < \frac{3k+5+\sqrt{9k^2+42k+33}}{6k+4}. \quad (1.10)$$

In particular, our result shows that Yagdjian's conjecture fails for  $\frac{3k+4}{3k+2} \leq p < p_{\text{crit}}(2k, 3)$ .

Next we discuss the global existence problem for (1.1). Denote by  $N = 1 + \frac{(m+2)n}{2}$  the homogeneous dimension of the operator  $\partial_t^2 - t^m \Delta$ . Then the exponent  $p$  leading to a conformally invariant equation in (1.1) is

$$p_{\text{conf}}(m, n) = \frac{N+2}{N-2} = \frac{(m+2)n+6}{(m+2)n-2}. \quad (1.11)$$

**Theorem 1.2** (Global existence for  $p > p_{\text{conf}}(m, n)$ ). *Let either  $p_{\text{conf}}(m, n) < p \leq \frac{(m+2)(n-2)+6}{(m+2)(n-2)-2}$  or  $p > \frac{(m+2)(n-2)+6}{(m+2)(n-2)-2}$  and  $p$  be an integer, where in the latter case the nonlinearity  $|u|^p$  is replaced with  $\pm u^p$ . Then there exists a constant  $\varepsilon_0 > 0$  such that problem (1.1) admits a global weak solution  $u \in L^r(\mathbb{R}_+^{n+1})$  whenever  $\|u_0\|_{H^s} + \|u_1\|_{H^{s-\frac{2}{m+2}}} \leq \varepsilon_0$ , where  $s = \frac{n}{2} - \frac{4}{(m+2)(p-1)}$  and  $r = \frac{(m+2)n+2}{4}(p-1)$ .*

**Remark 1.3.** It holds

$$p_{\text{conf}}(2k, 3) = \frac{3k+6}{3k+2} > \frac{3k+5+\sqrt{9k^2+42k+33}}{6k+4}.$$

So, we have especially improved the upper bound (from  $\min \left\{ \frac{3k+5}{3k+1}, \frac{5k+4}{3k+4} \right\}$  to  $\infty$ ) for the exponent  $p$  in Theorem B of Yagdjian to obtain global existence for small data solutions of problem (1.5).

**Remark 1.4.** If the initial data  $u_0 \in H^s(\mathbb{R}^n)$  and  $u_1 \in H^{s-\frac{2}{m+2}}(\mathbb{R}^n)$  with  $s \geq 0$  is given, then, by a scaling argument as in [9], one can deduce that problem (1.1) is ill-posed for  $s < \frac{n}{2} - \frac{4}{(m+2)(p-1)}$ . See [24] for details.

**Remark 1.5.** A direct verification shows  $p_{\text{crit}}(m, n) < p_{\text{conf}}(m, n)$  when  $n \geq 3$ . In a forthcoming paper [13] we shall establish the global existence of small data solution of (1.1) when  $p_{\text{crit}}(m, n) < p \leq p_{\text{conf}}(m, n)$ .

*Remark 1.6.* As in [18, page 368], where the semilinear wave equation (1.2) was studied, when  $p > \frac{(m+2)(n-2)+6}{(m+2)(n-2)-2}$ , we also impose additional restrictions on the exponent  $p$  and the nonlinearity appearing in (1.1). More specifically, we require that  $p$  is an integer and the nonlinearity is equal to  $\pm u^p$ .

There is an extensive list of results concerning the Cauchy problem for both linear and semilinear generalized Tricomi equations. For instances, for linear generalized Tricomi equations, Barros-Neto and Gelfand in [1] and Yagdjian in [29] computed the fundamental solution explicitly. More recently, the authors of [21–24] established the local existence as well as the singularity structure of low regularity solutions of the semilinear equation  $\partial_t^2 u - t^m \Delta u = f(t, x, u)$  in the degenerate hyperbolic region and the elliptic-hyperbolic mixed region, respectively, where  $f$  is a  $C^1$  function and has compact support with respect to the variable  $x$ . Yagdjian [30] obtained a number of interesting results about the global existence and the blowup of solutions of problem (1.1) when the exponent  $p$  belongs to a certain range. In [30], however, there is a gap between the global existence interval and the blowup interval; moreover, the critical exponent  $p_{\text{crit}}(m, n)$  was not determined there. In this paper and in a forthcoming paper [13], motivated by the Strauss conjecture, we will systematically study the blowup problem and the global existence problem for (1.1).

We now comment on the proofs of Theorem 1.1 and Theorem 1.2. To prove Theorem 1.1, we define the function  $G(t) = \int_{\mathbb{R}^n} u(t, x) dx$  as in [31] and, by applying some crucial techniques for the modified Bessel function as in [14, 23] and by choosing a good test function, we derive a Riccati-type ordinary differential inequality for  $G(t)$  by a delicate analysis of (1.1). From this, the blowup result in Theorem 1.1 can be derived under the positivity assumptions of  $u_0$  and  $u_1$ . To prove the global existence result in Theorem 1.2, motivated by [9, 18], where basic Strichartz estimates were obtained for the linear wave operator, we are required to establish Strichartz estimates for the generalized Tricomi operator  $\partial_t^2 - t^m \Delta$ . In this process, a series of inequalities are derived by applying an explicit formula for solutions of the linear generalized Tricomi equations and by utilizing some basic properties of related Fourier integral operators. Based on the resulting inequalities and the contraction mapping principle, we eventually complete the proof of Theorem 1.2.

This paper is organized as follows: In §2, the blowup result in Theorem 1.1 is obtained. In §3, some basic Strichartz inequalities are established for the linear generalized Tricomi operator  $\partial_t^2 - t^m \Delta$ . In §4, by the results in §3 and contractible mapping principle, we shall complete the proof of Theorem 1.2.

## 2 Proof of Theorem 1.1

In this section, we shall prove blowup in finite time for certain local solutions  $u$  of (1.1). To this end, we introduce the function  $G(t) = \int_{\mathbb{R}^n} u(t, x) dx$ . By some delicate analysis, we then obtain a Riccati-type differential inequality for  $G(t)$  so that blowup of  $G(t)$  can be deduced from the following result (see [26, Lemma 4]):

**Lemma 2.1.** *Suppose that  $G \in C^2([a, b]; \mathbb{R})$  and, for  $a \leq t < b$ ,*

$$G(t) \geq C_0(R + t)^\alpha, \quad (2.1)$$

$$G''(t) \geq C_1(R + t)^{-q} G(t)^p, \quad (2.2)$$

where  $C_0$ ,  $C_1$ , and  $R$  are some positive constants. Suppose further that  $p > 1$ ,  $\alpha \geq 1$ , and  $(p - 1)\alpha \geq q - 2$ . Then  $b$  is finite.

In view of  $\text{supp } u_i \subseteq B(0, M)$  ( $i = 0, 1$ ) and the finite propagation speed for solutions of hyperbolic equations, one has that, for any fixed  $t > 0$ , the support of  $u(t, \cdot)$  with respect to the variable  $x$  is contained in the ball  $B(0, M + \phi(t)) = \{x : |x| < M + \phi(t)\}$ , where  $\phi(t) = \frac{2}{m+2} t^{\frac{m+2}{2}}$ . Then it follows from an integration by parts that

$$G''(t) = \int_{\mathbb{R}^n} |u(t, x)|^p dx \geq \frac{\left| \int_{\mathbb{R}^n} u(t, x) dx \right|^p}{\left( \int_{|x| \leq M + \phi(t)} dx \right)^{p-1}} \geq C(M + t)^{-\frac{m+2}{2} n(p-1)} |G(t)|^p,$$

which means that  $G(t)$  fulfills inequality (2.2) with  $q = \frac{m+2}{2} n(p-1)$  (once inequality (2.1) has been verified demonstrating that  $G$  is positive). To establish (2.1), we introduce the following two functions: The first one is

$$\varphi(x) = \int_{\mathbb{S}^{n-1}} e^{x \cdot \omega} d\omega, \quad (2.3)$$

which was used in [31], where  $\varphi(x)$  is also shown to satisfy

$$\varphi(x) \sim C_n |x|^{-\frac{n-1}{2}} e^{|x|} \quad \text{as } |x| \rightarrow \infty. \quad (2.4)$$

The second function is the so-called modified Bessel function

$$K_\nu(t) = \int_0^\infty e^{-t \cosh z} \cosh(\nu z) dz, \quad \nu \in \mathbb{R},$$

which is a solution of the equation

$$\left( t^2 \frac{d^2}{dt^2} + t \frac{d}{dt} - (t^2 + \nu^2) \right) K_\nu(t) = 0, \quad t > 0.$$

From [7, page 24], we have

$$K_\nu(t) = \sqrt{\frac{\pi}{2t}} e^{-t} (1 + O(t^{-1})) \quad \text{as } t \rightarrow \infty, \quad (2.5)$$

provided that  $\text{Re } \nu > -1/2$ . Set

$$\lambda(t) = C_m t^{\frac{1}{2}} K_{\frac{1}{m+2}} \left( \frac{2}{m+2} t^{\frac{m+2}{2}} \right), \quad t > 0, \quad (2.6)$$

where the constant  $C_m > 0$  is chosen so that  $\lambda(t)$  satisfies

$$\begin{cases} \lambda''(t) - t^m \lambda(t) = 0, & t \geq 0 \\ \lambda(0) = 1, \quad \lambda(\infty) = 0. \end{cases} \quad (2.7)$$

Here is a list of properties of  $\lambda(t)$  (see [14, Lemma 2.1]):

**Lemma 2.2.** (i)  $\lambda(t)$  and  $-\lambda'(t)$  are both decreasing, moreover,  $\lim_{t \rightarrow \infty} \lambda(t) = \lim_{t \rightarrow \infty} \lambda'(t) = 0$ .

(ii) There exists a constant  $C > 1$  such that

$$\frac{1}{C} \leq \frac{|\lambda'(t)|}{\lambda(t) t^{\frac{m}{2}}} \quad \text{for } t > 0 \quad \text{and} \quad \frac{|\lambda'(t)|}{\lambda(t) t^{\frac{m}{2}}} \leq C \quad \text{for } t \geq 1. \quad (2.8)$$

We now introduce the test function  $\psi$  with

$$\psi(t, x) = \lambda(t)\varphi(x), \quad (2.9)$$

where the definition of  $\varphi$  has been given in (2.3). Let

$$G_1(t) = \int_{\mathbb{R}^n} u(t, x)\psi(t, x) dx. \quad (2.10)$$

Then

$$G''(t) = \int_{\mathbb{R}^n} |u(t, x)|^p dx \geq \frac{|G_1(t)|^p}{\left( \int_{|x| \leq M+\phi(t)} \psi(t, x)^{\frac{p}{p-1}} dx \right)^{p-1}}. \quad (2.11)$$

For the function  $G_1(t)$ , we have:

**Lemma 2.3.** *Under the assumptions of Theorem 1.1, there exists a  $t_0 > 0$  such that*

$$G_1(t) \geq C t^{-\frac{m}{2}}, \quad t \geq t_0. \quad (2.12)$$

*Proof.* In view of  $u \in C([0, T], H^1(\mathbb{R}^n))$ , one has that  $G_1(t)$  is a continuous function of  $t$ . In view of  $u_0 \geq 0$  and  $u_0 \not\equiv 0$ , we have

$$G_1(0) = \int_{\mathbb{R}^n} u_0(x)\varphi(x) dx \geq c_0,$$

where  $c_0$  is a positive constant. Hence, there exists a constant  $t_1 > 0$  such that, for  $0 \leq t \leq t_1$ ,

$$G_1(t) \geq \frac{c_0}{2}.$$

Similarly, by Lemma 2.2 (i) and  $u_1 \geq 0$  with  $u_1 \not\equiv 0$ , one can also choose a constant  $t_2 > 0$  such that, for  $0 \leq t \leq t_2$ ,

$$\int_{\mathbb{R}^n} \partial_t u(t, x)\psi(t, x) dx \geq \frac{c_0}{2} > 0.$$

Moreover, by the smoothness of  $\lambda(t)$  and  $\lambda(0) = 1$ , we can find a  $t_3 > 0$  such that

$$t_3^{\frac{m}{2}} \lambda(t_3) \geq c_1,$$

where  $c_1 > 0$  is some positive constant. Together with (i) and (ii) of Lemma 2.2, this yields, for  $0 \leq t \leq t_3$ ,

$$-\lambda'(t) \geq -\lambda'(t_3) = |\lambda'(t_3)| \geq C t_3^{\frac{m}{2}} \lambda(t_3) \geq C c_1.$$

Then, by the assumption that  $u_0 \geq 0$  but  $u_0 \not\equiv 0$ , we have that, for  $0 \leq t \leq t_3$ ,

$$\int_{\mathbb{R}^n} (-\partial_t \psi(t, x) u(t, x)) dx \geq \frac{c_2}{2} > 0,$$

where  $c_2$  is a positive constant. Note that

$$\Delta_x \left( \int_{\mathbb{S}^{n-1}} e^{x \cdot \omega} d\omega \right) = \int_{\mathbb{S}^{n-1}} \sum_{i=1}^n \omega_i^2 e^{x \cdot \omega} d\omega = \int_{\mathbb{S}^{n-1}} e^{x \cdot \omega} d\omega.$$

Let  $t_4 = \min\{t_1, t_2, t_3\} > 0$ . Then it follows from a direct computation that, for  $t > t_4$ ,

$$\begin{aligned} \int_{t_4}^t \int_{\mathbb{R}^n} |u|^p \psi \, dx \, ds &= \int_{t_4}^t \int_{\mathbb{R}^n} (\partial_s^2 u - s^m \Delta u) \psi \, dx \, ds \\ &= \int_{\mathbb{R}^n} (\psi \partial_s u - u \partial_s \psi) \, dx \Big|_{s=t} - \int_{\mathbb{R}^n} (\psi \partial_s u - u \partial_s \psi) \, dx \Big|_{s=t_4}, \end{aligned}$$

which leads to

$$\int_{\mathbb{R}^n} (\psi \partial_s u - u \partial_s \psi) \, dx \Big|_{s=t} \geq \int_{\mathbb{R}^n} (\psi \partial_s u - u \partial_s \psi) \, dx \Big|_{s=t_4} \geq c \equiv \frac{c_0}{2} + \frac{c_2}{2}.$$

This also yields

$$\begin{aligned} G_1'(t) - 2\lambda'(t) \int_{\mathbb{R}^n} u \varphi \, dx &= \frac{d}{dt} \left( \int_{\mathbb{R}^n} u \psi \, dx \right) - 2 \int_{\mathbb{R}^n} u \partial_t \psi \, dx \\ &= \int_{\mathbb{R}^n} (\psi \partial_s u - u \partial_s \psi) \, dx \Big|_{s=t} \geq c. \end{aligned} \tag{2.13}$$

Now assume that there is a constant  $t_5 > t_4$  such  $G_1(t_5) = 0$ , but  $G_1(t) > 0$  for  $t_4 \leq t < t_5$ . Then, for  $t_4 \leq t \leq t_5$ ,

$$\lambda(t) \int_{\mathbb{R}^n} u(t, x) \varphi(x) \, dx = \int_{\mathbb{R}^n} u(t, x) \psi(t, x) \, dx = G_1(t) \geq 0.$$

Together with Lemma 2.2 (i), this yields that for  $t_4 \leq t \leq t_5$ ,

$$\int_{\mathbb{R}^n} u(t, x) \varphi(x) \, dx \geq 0.$$

Furthermore, by Lemma 2.2 (ii), one has

$$-\lambda'(t) = |\lambda'(t)| \leq C\lambda(t)t^{\frac{m}{2}}.$$

Together with (2.17), this yields

$$G_1'(t) + Ct^{\frac{m}{2}} G_1(t) \geq G_1'(t) - 2\lambda'(t) \int_{\mathbb{R}^n} u \varphi \, dx \geq c. \tag{2.14}$$

Without loss of generality, we can assume that  $c = 1$  in (2.14). Then, by solving (2.14), we get that, for  $t_4 \leq t \leq t_5$ ,

$$e^{C\phi(t)} G_1(t) \geq e^{C\phi(t_4)} G_1(t_4) + \frac{t^{-\frac{m}{2}}}{C} \left( e^{C\phi(t)} - e^{C\phi(t_4)} \right). \tag{2.15}$$

Therefore,  $G_1(t_5) > 0$  holds which is a contradiction to  $G_1(t_5) = 0$ .

Thus, we have that, for all  $t \geq t_4$ ,

$$G_1(t) > 0.$$

Using Lemma 2.2 (ii) again and repeating the argument from above, one easily obtains the existence of a uniform positive constant  $\tilde{C}$  such that for  $t \geq t_4$

$$G_1(t) \geq \tilde{C} t^{-\frac{m}{2}}.$$

This proves Lemma 2.3. □



Relying on Lemma 2.3, we are now able to prove Theorem 1.1.

*Proof of Theorem 1.1.* By (2.5) and (2.6), we have that

$$\lambda(t) \sim t^{-\frac{m}{4}} e^{-\phi(t)} \quad \text{as } t \rightarrow \infty.$$

Next we estimate the denominator  $\left( \int_{|x| \leq M+\phi(t)} \psi(t, x)^{\frac{p}{p-1}} dx \right)^{p-1}$  in (2.11). Note that

$$\left( \int_{|x| \leq M+\phi(t)} \psi(t, x)^{\frac{p}{p-1}} dx \right)^{p-1} = \lambda(t)^p \left( \int_{|x| \leq M+\phi(t)} \varphi(x)^{\frac{p}{p-1}} dx \right)^{p-1}$$

and

$$|\varphi(x)| \leq C_n (1 + |x|)^{-\frac{n-1}{2}} e^{|x|}.$$

Then

$$\begin{aligned} & \int_{|x| \leq M+\phi(t)} \varphi(x)^{\frac{p}{p-1}} dx \\ & \leq C \int_0^{\frac{M+\phi(t)}{2}} (1+r)^{n-1-\frac{n-1}{2} \cdot \frac{p}{p-1}} e^{\frac{p}{p-1}r} dr + C \int_{\frac{M+\phi(t)}{2}}^{M+\phi(t)} (1+r)^{n-1-\frac{n-1}{2} \cdot \frac{p}{p-1}} e^{\frac{p}{p-1}r} dr \\ & \leq C e^{\frac{M+\phi(t)}{2}} + (M+\phi(t))^{n-1-\frac{n-1}{2} \cdot \frac{p}{p-1}} e^{p(M+\phi(t))} \\ & \leq C (M+\phi(t))^{n-1-\frac{n-1}{2} \cdot \frac{p}{p-1}} e^{p(M+\phi(t))} \end{aligned}$$

and

$$\begin{aligned} \left( \int_{|x| \leq M+\phi(t)} \psi(t, x)^{\frac{p}{p-1}} dx \right)^{p-1} & \leq C t^{-\frac{m}{4}p} e^{-p\phi(t)} (M+\phi(t))^{(n-1)(p-1)-\frac{n-1}{2}p} e^{p(M+\phi(t))} \\ & \leq C t^{-\frac{m}{4}p} (M+\phi(t))^{(n-1)(p-1)-\frac{n-1}{2}p}. \end{aligned} \tag{2.16}$$

Therefore, it follows from (2.11) and (2.16) that, for  $t \geq t_0$ ,

$$G''(t) \geq c t^{-\frac{m}{4}p} (M+\phi(t))^{\frac{n-1}{2}p-(n-1)(p-1)} \geq C t^{\frac{p}{2}} (M+\phi(t))^{n-1-\frac{n}{2}p}. \tag{2.17}$$

Integrating (2.17) twice gives

$$G(t) \geq C (M+t)^{\frac{p}{2}+2+\frac{m+2}{2}(n-1-\frac{n}{2}p)} + C_1 (t-t_0) + C_2.$$

Note that if

$$\frac{p}{2} + 2 + \frac{m+2}{2} \left( n-1-\frac{n}{2}p \right) > 1 \tag{2.18}$$

holds, then one has, for  $t \geq t_0$ ,

$$G(t) \geq C (M+t)^{\frac{p}{2}+2+\frac{m+2}{2}(n-1-\frac{n}{2}p)}. \tag{2.19}$$

This means that condition (2.1) holds with  $\alpha = \frac{p}{2} + 2 + \frac{m+2}{2}(n-1-\frac{n}{2}p)$ .

To conclude the proof of Theorem 1.1 we now apply Lemma 2.1. For  $n \geq 3$ , one easily checks that all  $p < p_{\text{conf}}(m, n)$  satisfy (2.18). On the other hand, if we take

$$\alpha = \frac{p}{2} + 2 + \frac{m+2}{2} \left( n - 1 - \frac{n}{2} p \right), \quad q = \frac{m+2}{2} n (p - 1),$$

then the condition  $(p-1)\alpha > q-2$  in Lemma 2.1 becomes

$$(p-1) \left( \frac{p}{2} + 2 + \frac{m+2}{2} \left( n - 1 - \frac{n}{2} p \right) \right) > \frac{m+2}{2} n (p-1) - 2,$$

which is equivalent to

$$\left( (m+2) \frac{n}{2} - 1 \right) p^2 + \left( (m+2) \left( 1 - \frac{n}{2} \right) - 3 \right) p - (m+2) < 0.$$

The latter means that

$$p < p_{\text{crit}}(m, n) = \frac{\left( \frac{n}{2} - 1 \right) (m+2) + 3 + \sqrt{\left( \frac{n^2}{4} + n + 1 \right) (m+2)^2 + (3n-10)(m+2) + 9}}{(m+2)n-2}.$$

By a direct verification, we have that  $p_{\text{crit}}(m, n)$  satisfies (1.9) and that  $p_{\text{crit}}(m, n) < p_{\text{conf}}(m, n)$  holds.

We complete the proof of Theorem 1.1 by appealing to Lemma 2.1 with  $a = t_0$  and  $b = t$ .  $\square$

### 3 Strichartz estimates for the generalized Tricomi operator

Before establishing Strichartz estimates for the generalized Tricomi operator, we recall two results from [10, Lemma 3.8] and [2, Theorem 1.2].

**Lemma 3.1.** *Let  $\beta \in C_0^\infty((1/2, 2))$  and  $\sum_{j=-\infty}^{\infty} \beta(2^{-j}\tau) \equiv 1$  for  $\tau > 0$ . Define the Littlewood-Paley operators as*

$$G_j(t, x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \beta(2^{-j}|\xi|) \hat{G}(t, \xi) d\xi, \quad j \in \mathbb{Z}.$$

Then

$$\|G\|_{L_t^s L_x^q} \leq C \left( \sum_{j=-\infty}^{\infty} \|G_j\|_{L_t^s L_x^q}^2 \right)^{1/2}, \quad 2 \leq q < \infty, 2 \leq s \leq \infty,$$

and

$$\left( \sum_{j=-\infty}^{\infty} \|G_j\|_{L_t^r L_x^p}^2 \right)^{1/2} \leq C \|G\|_{L_t^r L_x^p}, \quad 1 < p \leq 2, 1 \leq r \leq 2.$$

**Lemma 3.2.** *Suppose that  $1 \leq p < q \leq \infty$ . Let  $T : L^p(\mathbb{R}) \rightarrow L^q(\mathbb{R})$  be a bounded linear operator which is defined by*

$$Tf(x) = \int_{\mathbb{R}} K(x, y) f(y) dy,$$

where  $K$  is locally integrable. Define

$$\tilde{T}f(x) = \int_{-\infty}^x K(x, y) f(y) dy.$$

Then

$$\|\tilde{T}f\|_{L^q} \leq C_{p,q} \|T\|_{L^p \rightarrow L^q} \|f\|_{L^p}.$$

In order to prove Theorem 1.2, we need to establish Strichartz estimates for the operator  $\partial_t^2 - t^m \Delta$ . To this end, we study the linear Cauchy problem

$$\begin{cases} \partial_t^2 u - t^m \Delta u = F(t, x), & (t, x) \in \mathbb{R}_+^{n+1}, \\ u(0, \cdot) = f(x), \quad \partial_t u(0, \cdot) = g(x). \end{cases} \quad (3.1)$$

Note that the solution  $u$  of (3.1) can be written as

$$u(t, x) = v(t, x) + w(t, x),$$

where  $v$  solves the homogeneous problem

$$\begin{cases} \partial_t^2 v - t^m \Delta v = 0, & (t, x) \in \mathbb{R}_+^{n+1}, \\ v(0, \cdot) = f(x), \quad \partial_t v(0, \cdot) = g(x). \end{cases} \quad (3.2)$$

and  $w$  solves the inhomogeneous problem with zero initial data

$$\begin{cases} \partial_t^2 w - t^m \Delta w = F(t, x), & (t, x) \in \mathbb{R}_+^{n+1}, \\ w(0, \cdot) = 0, \quad \partial_t w(0, \cdot) = 0. \end{cases} \quad (3.3)$$

Let  $\dot{H}^s(\mathbb{R}^n)$  denote the homogeneous Sobolev space with norm

$$\|f\|_{\dot{H}^s(\mathbb{R}^n)} = \| |D_x|^s f \|_{L^2(\mathbb{R}^n)},$$

where

$$|D_x| = \sqrt{-\Delta}.$$

If  $g \equiv 0$  in (3.2), we intend to establish the Strichartz-type inequality

$$\|v\|_{L_t^q L_x^r} \leq C \|f\|_{\dot{H}^s(\mathbb{R}^n)},$$

where  $q \geq 1$  and  $r \geq 1$  are suitable constants related to  $s$ . One obtains by a scaling argument that those indices should satisfy

$$\frac{1}{q} + \frac{m+2}{2} \cdot \frac{n}{r} = \frac{m+2}{2} \left( \frac{n}{2} - s \right). \quad (3.4)$$

Setting  $r = q$  and  $s = \frac{1}{m+2}$  in (3.4), we find that

$$q = q_0 \equiv \frac{2((m+2)n+2)}{(m+2)n-2} > 1, \quad n \geq 2, m \in \mathbb{N}. \quad (3.5)$$

Note that problem (1.1) is ill-posed for  $u_0 \in H^s(\mathbb{R}^n)$  with  $s < \frac{n}{2} - \frac{4}{(m+2)(p-1)}$  (see Remark 1.4), while  $p \geq p_{\text{conf}}(m, n)$  and  $s = \frac{n}{2} - \frac{4}{(m+2)(p-1)}$  imply  $s \geq \frac{1}{m+2}$ .

We now prove:

**Lemma 3.3.** *Let  $n \geq 2$  and  $v$  solve problem (3.2). Further let  $\frac{1}{m+2} \leq s < \frac{n}{2}$ . Then*

$$\|v\|_{L^q(\mathbb{R}_+^{n+1})} \leq C \left( \|f\|_{\dot{H}^s(\mathbb{R}^n)} + \|g\|_{\dot{H}^{s-\frac{2}{m+2}}(\mathbb{R}^n)} \right), \quad (3.6)$$

where  $q = \frac{2((m+2)n+2)}{(m+2)(n-2s)} \geq q_0$  and the constant  $C > 0$  only depends on  $m, n$ , and  $s$ .

*Proof.* It follows from [30] that the solution  $v$  of (3.2) can be written as

$$v(t, x) = V_1(t, D_x)f(x) + V_2(t, D_x)g(x),$$

where the operators  $V_j(t, D_x)$  ( $j = 1, 2$ ) have the symbols  $V_j(t, \xi)$  given by

$$\begin{aligned} V_1(t, \xi) = V_1(t, |\xi|) &= \frac{\Gamma(\frac{m}{m+2})}{\Gamma(\frac{m}{2(m+2)})} e^{\frac{z}{2}} H_+ \left( \frac{m}{2(m+2)}, \frac{m}{m+2}; z \right) \\ &\quad + \frac{\Gamma(\frac{m}{m+2})}{\Gamma(\frac{m}{2(m+2)})} e^{-\frac{z}{2}} H_- \left( \frac{m}{2(m+2)}, \frac{m}{m+2}; z \right) \end{aligned} \quad (3.7)$$

and

$$\begin{aligned} V_2(t, \xi) = V_2(t, |\xi|) &= t \frac{\Gamma(\frac{m+4}{m+2})}{\Gamma(\frac{m+4}{2(m+2)})} e^{\frac{z}{2}} H_+ \left( \frac{m+4}{2(m+2)}, \frac{m+4}{m+2}; z \right) \\ &\quad + t \frac{\Gamma(\frac{m+4}{m+2})}{\Gamma(\frac{m+4}{2(m+2)})} e^{-\frac{z}{2}} H_- \left( \frac{m+4}{2(m+2)}, \frac{m+4}{m+2}; z \right), \end{aligned} \quad (3.8)$$

where  $z = 2i\phi(t)|\xi|$ . For  $\alpha, \nu \in \mathbb{R}, \omega \in \mathbb{C}$ , we have

$$\begin{aligned} H_+(\alpha, \nu; \omega) &= \frac{e^{-i\pi(\nu-\alpha)}}{e^{i\pi(\nu-\alpha)} - e^{-i\pi(\nu-\alpha)}} \frac{1}{\Gamma(\nu-\alpha)} \omega^{\alpha-\nu} \int_{\infty}^{(0+)} e^{-\theta} \theta^{\nu-\alpha-1} \left(1 - \frac{\theta}{\omega}\right)^{\alpha-1} d\theta, \\ H_-(\alpha, \nu; \omega) &= \frac{1}{e^{i\pi\alpha} - e^{-i\pi\alpha}} \frac{1}{\Gamma(\alpha)} \omega^{-\alpha} \int_{\infty}^{(0+)} e^{-\theta} \theta^{\alpha-1} \left(1 + \frac{\theta}{\omega}\right)^{\nu-\alpha-1} d\theta. \end{aligned}$$

By [30, Section 3], one has that, for  $\phi(t)|\xi| \geq 1$ ,

$$|\partial_{\xi}^{\beta} H_+(\alpha, \gamma; 2i\phi(t)|\xi|)| \leq C (\phi(t)|\xi|)^{\alpha-\gamma} (1 + |\xi|^2)^{-\frac{|\beta|}{2}}, \quad (3.9)$$

$$|\partial_{\xi}^{\beta} H_-(\alpha, \gamma; 2i\phi(t)|\xi|)| \leq C (\phi(t)|\xi|)^{-\alpha} (1 + |\xi|^2)^{-\frac{|\beta|}{2}}. \quad (3.10)$$

We only estimate  $V_1$ , since estimating  $V_2$  is similar. Indeed, up to a factor of  $t\phi(t)^{-\frac{m+4}{2(m+2)}} = C_m\phi(t)^{-\frac{m}{2(m+2)}}$ , the powers of  $t$  appearing in  $V_1$  or  $V_2$  are the same.

Choose  $\chi \in C^\infty(\mathbb{R}_+)$  such that

$$\chi(s) = \begin{cases} 1, & s \geq 2, \\ 0, & s \leq 1. \end{cases} \quad (3.11)$$

Then

$$\begin{aligned} V_1(t, |\xi|)\hat{f}(\xi) &= \chi(\phi(t)|\xi|)V_1(t, |\xi|)\hat{f}(\xi) + (1 - \chi(\phi(t)|\xi|))V_1(t, |\xi|)\hat{f}(\xi) \\ &\equiv \hat{v}_1(t, \xi) + \hat{v}_2(t, \xi). \end{aligned} \quad (3.12)$$

Using (3.7), (3.9), and (3.10), we derive that

$$v_1(t, x) = C_m \left( \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \phi(t)|\xi|)} a_{11}(t, \xi) \hat{f}(\xi) d\xi + \int_{\mathbb{R}^n} e^{i(x \cdot \xi - \phi(t)|\xi|)} a_{12}(t, \xi) \hat{f}(\xi) d\xi \right), \quad (3.13)$$

where  $C_m > 0$  is a constant only depending on  $m$ , and, for  $l = 1, 2$ ,

$$|\partial_\xi^\beta a_{1l}(t, \xi)| \leq C_{l\beta} |\xi|^{-|\beta|} (\phi(t)|\xi|)^{-\frac{m}{2(m+2)}}.$$

On the other hand, it follows from [6] that

$$V_1(t, |\xi|) = e^{-\frac{z}{2}} \Phi \left( \frac{m}{2(m+2)}, \frac{m}{m+2}; z \right), \quad (3.14)$$

where  $\Phi$  is the confluent hypergeometric functions which is analytic with respect to the variable  $z = 2i\phi(t)|\xi|$ . Then

$$|\partial_\xi((1 - \chi(\phi(t)|\xi|))V_1(t, |\xi|))| \leq C(1 + \phi(t)|\xi|)^{-\frac{m}{2(m+2)}} |\xi|^{-1}.$$

Similarly, one has

$$|\partial_\xi^\beta((1 - \chi(\phi(t)|\xi|))V_1(t, |\xi|))| \leq C(1 + \phi(t)|\xi|)^{-\frac{m}{2(m+2)}} |\xi|^{-|\beta|}.$$

Thus, we arrive at

$$v_2(t, x) = C_m \left( \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \phi(t)|\xi|)} a_{21}(t, \xi) \hat{f}(\xi) d\xi + \int_{\mathbb{R}^n} e^{i(x \cdot \xi - \phi(t)|\xi|)} a_{22}(t, \xi) \hat{f}(\xi) d\xi \right), \quad (3.15)$$

where, for  $l = 1, 2$ ,

$$|\partial_\xi^\beta a_{2l}(t, \xi)| \leq C_{l\beta} (1 + \phi(t)|\xi|)^{-\frac{m}{2(m+2)}} |\xi|^{-|\beta|}.$$

Substituting (3.13) and (3.15) into (3.12) yields

$$V_1(t, D_x)f(x) = C_m \left( \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \phi(t)|\xi|)} a_1(t, \xi) \hat{f}(\xi) d\xi + \int_{\mathbb{R}^n} e^{i(x \cdot \xi - \phi(t)|\xi|)} a_2(t, \xi) \hat{f}(\xi) d\xi \right),$$

where the  $a_l$  ( $l = 1, 2$ ) satisfy

$$|\partial_\xi^\beta a_l(t, \xi)| \leq C_{l\beta} (1 + \phi(t)|\xi|)^{-\frac{m}{2(m+2)}} |\xi|^{-|\beta|}. \quad (3.16)$$

We only treat the integral  $\int_{\mathbb{R}^n} e^{i(x \cdot \xi + \phi(t)|\xi|)} a_1(t, \xi) \hat{f}(\xi) d\xi$ , since the treatment of the integral  $\int_{\mathbb{R}^n} e^{i(x \cdot \xi - \phi(t)|\xi|)} a_2(t, \xi) \hat{f}(\xi) d\xi$  is similar. Denote

$$(Af)(t, x) = \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \phi(t)|\xi|)} a_1(t, \xi) \hat{f}(\xi) d\xi. \quad (3.17)$$

We will show that

$$\|(Af)(t, x)\|_{L^q(\mathbb{R}_+^{n+1})} \leq C \|f\|_{\dot{H}^s(\mathbb{R}^n)}. \quad (3.18)$$

Note that if we set

$$\tilde{a}(t, \xi) = \frac{a_1(t, \xi)}{|\xi|^s}, \quad \hat{h}(\xi) = |\xi|^s \hat{f}(\xi),$$

then (3.18) is equivalent to

$$\left\| \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \phi(t)|\xi|)} \tilde{a}(t, \xi) \hat{h}(\xi) d\xi \right\|_{L^q(\mathbb{R}_+^{n+1})} \leq C \|h\|_{L^2(\mathbb{R}^n)}. \quad (3.19)$$

We denote the integral operator in the left-hand side of (3.19) still by  $A$ . In order to prove (3.19) it suffices to establish the its dual version

$$\|A^*G\|_{L^2(\mathbb{R}^n)} \leq C \|G\|_{L^p(\mathbb{R}_+^{n+1})}, \quad (3.20)$$

where

$$(A^*G)(y) = \int_{\mathbb{R}^n} \int_{\mathbb{R}_+^{n+1}} e^{i(y-x) \cdot \xi - \phi(t)|\xi|} \overline{\tilde{a}(t, \xi)} G(t, x) dt dx d\xi$$

is the adjoint operator of  $A$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ , and  $1 \leq p \leq p_0 \equiv \frac{2((m+2)n+2)}{(m+2)n+6}$  (note that  $\frac{1}{p_0} + \frac{1}{q_0} = 1$ ). In view of

$$\int_{\mathbb{R}^n} |(A^*G)(y)|^2 dy = \int_{\mathbb{R}_+^{n+1}} (AA^*G)(t, x) \overline{G(t, x)} dt dx \leq \|AA^*G\|_{L^q(\mathbb{R}_+^{n+1})} \|G\|_{L^p(\mathbb{R}_+^{n+1})}, \quad (3.21)$$

one derives that (3.20) holds if

$$\|AA^*G\|_{L^q(\mathbb{R}_+^{n+1})} \leq C \|G\|_{L^p(\mathbb{R}_+^{n+1})}, \quad 1 \leq p \leq p_0. \quad (3.22)$$

One can write

$$(AA^*G)(t, x) = \int_{\mathbb{R}_+^{n+1}} \int_{\mathbb{R}^n} e^{i((\phi(t)-\phi(\tau))|\xi|+(x-y) \cdot \xi)} \tilde{a}(t, \xi) \overline{\tilde{a}(\tau, \xi)} G(\tau, y) d\xi d\tau dy. \quad (3.23)$$

If we choose a function  $\beta \in C_0^\infty((1/2, 2))$  as in Lemma 3.1 and set  $a_\lambda(t, \tau, \xi) = \beta(|\xi|/\lambda) \tilde{a}(t, \xi) \overline{\tilde{a}(\tau, \xi)}$  for  $\lambda > 0$ , then we obtain a dyadic decomposition of the operator  $AA^*$  by

$$(AA^*)_\lambda G = \int_{\mathbb{R}_+^{n+1}} \int_{\mathbb{R}^n} e^{i((\phi(t)-\phi(\tau))|\xi|+(x-y) \cdot \xi)} a_\lambda(t, \tau, \xi) G(\tau, y) d\xi d\tau dy. \quad (3.24)$$

In order to prove (3.22), we only need to prove

$$\|(AA^*)_\lambda G\|_{L^{p'}(\mathbb{R}_+^{n+1})} \leq C \|G\|_{L^p(\mathbb{R}_+^{n+1})}, \quad 1 \leq p \leq p_0, \quad (3.25)$$

with the constant  $C > 0$  independent of  $\lambda > 0$ . Indeed, if (3.25) holds, then it follows from Lemma 3.1 and  $p \leq p_0 = \frac{2((m+2)n+2)}{(m+2)n+6} < 2$  that

$$\begin{aligned} \|AA^*G\|_{L^q}^2 &\leq C \sum_{j \in \mathbb{Z}} \|(AA^*)_{2^j} G\|_{L^q}^2 \leq C \sum_{j \in \mathbb{Z}} \sum_{k: |j-k| \leq C_0} \|(AA^*)_{2^j} G_k\|_{L^q}^2 \\ &\leq C \sum_{j \in \mathbb{Z}} \sum_{k: |j-k| \leq C_0} \|G_k\|_{L^p}^2 \leq C \|G\|_{L^p(\mathbb{R}_+^{n+1})}^2, \end{aligned}$$

where  $\hat{G}_k(\tau, \xi) = \beta(2^{-k}|\xi|) \hat{G}(\tau, \xi)$ .

Next we prove (3.25). We will use interpolation between the two cases  $p = 1$  and  $p = p_0$ .

For  $p = 1$ , a direct analysis shows that

$$|a_\lambda(t, \tau, \xi)| \leq |\xi|^{-2s}$$

and

$$\begin{aligned} \|(AA^*)_\lambda G\|_{L^\infty(\mathbb{R}_+^{n+1})} &\leq \int_{\mathbb{R}_+^{n+1}} \left| \int_{\mathbb{R}^n} e^{i[(\phi(t)-\phi(\tau))|\xi|+(x-y)\cdot\xi]} a_\lambda(t, \tau, \xi) d\xi \right| |G(\tau, y)| dy d\tau \\ &\leq \int_{\mathbb{R}_+^{n+1}} \left| \int_{\mathbb{R}^n} \beta\left(\frac{|\xi|}{\lambda}\right) |\xi|^{-2s} d\xi \right| |G(\tau, y)| dy d\tau \\ &\leq C \lambda^{n-2s} \|G\|_{L^1(\mathbb{R}_+^{n+1})}. \end{aligned} \tag{3.26}$$

Next we prove the endpoint case  $p = p_0$  in (3.25). Namely, we shall show that

$$\|(AA^*)_\lambda G\|_{L^q(\mathbb{R}_+^{n+1})} \leq C \lambda^{\frac{2}{m+2}-2s} \|G\|_{L^{p_0}(\mathbb{R}_+^{n+1})}. \tag{3.27}$$

Note that, for any  $t, \tau \in \mathbb{R}_+$  and  $\bar{t} = \max\{t, \tau\}$ , one has that

$$\left| \partial_\xi^\beta \left( \bar{t}^{\frac{m}{(m+2)n+2}} a_\lambda(t, \tau, \xi) \right) \right| \leq |\xi|^{-2s - \frac{2m}{(m+2)((m+2)n+2)} - |\beta|}. \tag{3.28}$$

Indeed, without loss of generality, one can assume that  $t \geq \tau$ . Then it follows from (3.16) and a direct computation that

$$\begin{aligned} \left| \partial_\xi^\beta \left( \bar{t}^{\frac{m}{(m+2)n+2}} a_\lambda(t, \tau, \xi) \right) \right| &\leq \bar{t}^{\frac{m}{(m+2)n+2}} (1 + \phi(t)|\xi|)^{-\frac{m}{2(m+2)}} (1 + \phi(\tau)|\xi|)^{-\frac{m}{2(m+2)}} |\xi|^{-|\beta|-2s} \\ &\leq \phi(t)^{\frac{2m}{(m+2)((m+2)n+2)}} (\phi(t)|\xi|)^{-\frac{2m}{(m+2)((m+2)n+2)}} |\xi|^{-|\beta|-2s} \\ &\leq |\xi|^{-2s - \frac{2m}{(m+2)((m+2)n+2)} - |\beta|}. \end{aligned}$$

Set

$$b(t, \tau, \xi) = \lambda^{2s + \frac{2m}{(m+2)((m+2)n+2)}} \bar{t}^{\frac{m}{(m+2)n+2}} a_\lambda(t, \tau, \xi).$$

Then

$$|\partial_\xi^\beta b(t, \tau, \xi)| \leq |\xi|^{-|\beta|}$$

and we can write

$$(AA^*)_\lambda G = \int_{\mathbb{R}_+^{n+1}} \int_{\mathbb{R}^n} e^{i[(\phi(t)-\phi(\tau))|\xi|+(x-y)\cdot\xi]} \bar{t}^{-\frac{m}{(m+2)n+2}} \lambda^{-2s - \frac{2m}{(m+2)((m+2)n+2)}} G(\tau, y) dy d\tau$$

$$\times b(t, \tau, \xi) G(\tau, y) d\xi dy d\tau.$$

Introduce the operator

$$T_{t,\tau} f(x) = \int \int e^{i((\phi(t)-\phi(\tau))|\xi|+(x-y)\cdot\xi)} \bar{t}^{-\frac{m}{(m+2)n+2}} b(t, \tau, \xi) f(y) d\xi dy.$$

Then, by  $\max\{t, \tau\} \geq |t - \tau|$ , we have that

$$\|T_{t,\tau} f\|_{L^2(\mathbb{R}^n)} \leq C |t - \tau|^{-\frac{m}{(m+2)n+2}} \|f\|_{L^2(\mathbb{R}^n)}. \quad (3.29)$$

On the other hand, it follows from the method of stationary phase that

$$\begin{aligned} \|T_{t,\tau} f\|_{L^\infty(\mathbb{R}^n)} &\leq C \lambda^{\frac{n+1}{2}} \bar{t}^{-\frac{m}{(m+2)n+2}} |\phi(t) - \phi(\tau)|^{-\frac{n-1}{2}} \|f\|_{L^1(\mathbb{R}^n)} \\ &\leq C \lambda^{\frac{n+1}{2}} |t - \tau|^{-\frac{m}{(m+2)n+2}} |t - \tau|^{-\frac{n-1}{2} \cdot \frac{m+2}{2}} \|f\|_{L^1(\mathbb{R}^n)}. \end{aligned} \quad (3.30)$$

Together with (3.29), this yields

$$\|T_{t,\tau} f\|_{L^{q_0}(\mathbb{R}^n)} \leq C \lambda^{\frac{2(n+1)}{(m+2)n+2}} |t - \tau|^{-\frac{(m+2)n-2}{(m+2)n+2}} \|f\|_{L^{p_0}(\mathbb{R}^n)}. \quad (3.31)$$

Because of  $1 - (\frac{1}{p_0} - \frac{1}{q_0}) = \frac{(m+2)n-2}{(m+2)n+2}$ , it follows from the Hardy-Littlewood-Sobolev inequality that

$$\begin{aligned} \|(AA^*)_\lambda G\|_{L^{q_0}(\mathbb{R}_+^{n+1})} &= \left\| \int_0^\infty T_{t,\tau} G d\tau \right\|_{L^{q_0}(\mathbb{R}_+^{n+1})} \\ &\leq C \lambda^{-2s - \frac{2m}{(m+2)((m+2)n+2)}} \lambda^{\frac{2(n+1)}{(m+2)n+2}} \left\| \int_{\mathbb{R}} |t - \tau|^{-\frac{(m+2)n-2}{(m+2)n+2}} \|G(\tau, \cdot)\|_{L^{p_0}(\mathbb{R}^n)} d\tau \right\|_{L^{p_0}(\mathbb{R})} \\ &\leq C \lambda^{-2s + \frac{2}{m+2}} \|G\|_{L^{p_0}(\mathbb{R}_+^{n+1})}. \end{aligned} \quad (3.32)$$

By interpolation between (3.26) and (3.32), we have that, for  $1 \leq p \leq p_0$ ,

$$\|(AA^*)_\lambda G\|_{L^q(\mathbb{R}_+^{n+1})} \leq C \lambda^{-2s + 2\left(\frac{n}{2} - \frac{(m+2)n+2}{(m+2)q}\right)} \|G\|_{L^p(\mathbb{R}_+^{n+1})}.$$

In particular, choosing  $s = \frac{n}{2} - \frac{(m+2)n+2}{(m+2)q}$  yields estimate (3.18) for  $v_1(t, x)$ . The same estimate for  $v_2(t, x)$  is analogously obtained.

Thus, the proof of Lemma 3.3 is complete.  $\square$

Next we treat the inhomogeneous problem (3.3). Based on Lemmas 3.1 and 3.2, we establish the following estimate:

**Lemma 3.4.** *Let  $n \geq 2$  and  $w$  solve (3.3). Then*

$$\|w\|_{L^q(\mathbb{R}_+^{n+1})} \leq C \left\| |D_x|^{\gamma - \frac{1}{m+2}} F \right\|_{L^{p_0}(\mathbb{R}_+^{n+1})}, \quad (3.33)$$

where  $\gamma = \frac{n}{2} - \frac{(m+2)n+2}{q(m+2)}$ ,  $q_0 \leq q < \infty$ , and the constant  $C > 0$  only depends on  $m, n$  and  $q$ .



*Proof.* It follows from problem (3.3) that

$$w(t, x) = \int_0^t (V_2(t, D_x)V_1(\tau, D_x) - V_1(t, D_x)V_2(\tau, D_x)) F(\tau, x) d\tau.$$

To estimate  $w(t, x)$ , it suffices to treat the term  $\int_0^t V_2(t, D_x)V_1(\tau, D_x)F(\tau, x)d\tau$  since the treatment on the term  $\int_0^t V_1(t, D_x)V_2(\tau, D_x)F(\tau, x)d\tau$  is completely analogous. Choose a cut-off function  $\chi$  as in (3.11). Set

$$\begin{aligned} w_1(t, x) &= \int_0^t \chi(\phi(t)D_x)\chi(\phi(\tau)D_x)V_2(t, D_x)V_1(\tau, D_x)F(\tau, x) d\tau, \\ w_2(t, x) &= \int_0^t \chi(\phi(t)D_x)(1 - \chi(\phi(\tau)D_x))V_2(t, D_x)V_1(\tau, D_x)F(\tau, x) d\tau, \\ w_3(t, x) &= \int_0^t (1 - \chi(\phi(t)D_x))\chi(\phi(\tau)D_x)V_2(t, D_x)V_1(\tau, D_x)F(\tau, x) d\tau, \\ w_4(t, x) &= \int_0^t (1 - \chi(\phi(t)D_x))(1 - \chi(\phi(\tau)D_x))V_2(t, D_x)V_1(\tau, D_x)F(\tau, x) d\tau. \end{aligned}$$

Together with (3.7)-(3.10), as in the proof of Lemma 3.3, we can write  $\sum_{j=1}^4 w_j$  as

$$\sum_{j=1}^4 w_j = (AF)(t, x) \equiv \int_0^t \int_{\mathbb{R}^n} e^{i(x \cdot \xi + (\phi(t) - \phi(\tau))|\xi|)} a(t, \tau, \xi) \hat{F}(\tau, \xi) d\xi d\tau, \quad (3.34)$$

where  $a(t, \tau, \xi)$  satisfies

$$|\partial_\xi^\beta a(t, \xi)| \leq C (1 + \phi(t)|\xi|)^{-\frac{m}{2(m+2)}} (1 + \phi(\tau)|\xi|)^{-\frac{m}{2(m+2)}} |\xi|^{-\frac{2}{m+2} - |\beta|}.$$

To treat  $(AF)(t, x)$  conveniently, we introduce the more general operator

$$(A^\alpha F)(t, x) = \int_0^t \int_{\mathbb{R}^n} e^{i(x \cdot \xi + (\phi(t) - \phi(\tau))|\xi|)} a(t, \tau, \xi) \hat{F}(\tau, \xi) \frac{d\xi}{|\xi|^\alpha} d\tau, \quad (3.35)$$

where  $0 \leq \alpha < \frac{n}{2}$  is a parameter.

As in the proof of Lemma 3.3, we shall use the Littlewood-Paley argument with a bump function  $\beta$  as in Lemma 3.1. Define the operator

$$A_j^\alpha F(t, x) = \int_0^t \int_{\mathbb{R}^n} e^{i(x \cdot \xi + (\phi(t) - \phi(\tau))|\xi|)} \beta\left(\frac{|\xi|}{2^j}\right) a(t, \tau, \xi) \hat{F}(\tau, \xi) \frac{d\xi}{|\xi|^\alpha} d\tau. \quad (3.36)$$

Note that our aim is to establish the inequality, for  $\gamma = \frac{n}{2} - \frac{(m+2)n+2}{q(m+2)}$ ,  $q_0 \leq q < \infty$ ,

$$\|w\|_{L^q(\mathbb{R}_+^{n+1})} \leq C \| |D_x|^{\gamma - \frac{1}{m+2}} F \|_{L^{p_0}}$$

which is equivalent to proving that

$$\left\| |D_x|^{-\gamma + \frac{1}{m+2}} w \right\|_{L^q(\mathbb{R}_+^{n+1})} \leq C \|F\|_{L^{p_0}(\mathbb{R}_+^{n+1})}.$$

In terms of the operator  $A^\alpha$  in (3.35) with  $\alpha = \gamma - \frac{1}{m+2}$ , it suffices to establish

$$\|A^\alpha F\|_{L^q(\mathbb{R}_+^{n+1})} \leq C \|F\|_{L^{p_0}(\mathbb{R}_+^{n+1})}. \quad (3.37)$$

in order to complete the proof of (3.33).

Note that  $p_0 < 2 < q < \infty$ . To derive (3.37), it follows from Lemma 3.1 that we only need to prove

$$\|A_j^\alpha F\|_{L^q(\mathbb{R}_+^{n+1})} \leq C \|F\|_{L^{p_0}(\mathbb{R}_+^{n+1})}. \quad (3.38)$$

By interpolation, it suffices to prove that (3.38) holds for the special cases  $q = q_0$  and  $q = \infty$ . Denote the corresponding indices  $\alpha$  by  $\alpha_0$  and  $\alpha_1$ . A direct computation yields  $\alpha_0 = \frac{n}{2} - \frac{(m+2)n+2}{q_0(m+2)} - \frac{1}{m+2} = 0$  and  $\alpha_1 = \frac{n}{2} - \frac{1}{m+2}$ . We now treat  $A_j^{\alpha_0} = A_j^0$ . Let

$$T_j^0 G(t, \tau, x) = \int_{\mathbb{R}^n} e^{i(x \cdot \xi + (\phi(t) - \phi(\tau))|\xi|)} \beta\left(\frac{|\xi|}{2^j}\right) a(t, \tau, \xi) \hat{G}(\tau, \xi) d\xi.$$

We can repeat the derivation of (3.31) to get

$$\|T_j^0 G(t, \tau, \cdot)\|_{L^{p'_0}(\mathbb{R}^n)} \leq C |t - \tau|^{-\frac{(m+2)n-2}{(m+2)n+2}} \|G(\tau, \cdot)\|_{L^{p_0}}. \quad (3.39)$$

Note that  $A_j^0 G(t, x) = \int_0^t T_j^0 G(t, \tau, x) d\tau$ . Then, by (3.39) and the Hardy-Littlewood-Sobolev inequality, we get

$$\left\| \int_{\mathbb{R}} \|T_j^0 G(t, \tau, x)\|_{L_x^{q_0}} d\tau \right\|_{L_t^{q_0}} \leq C \|G\|_{L^{p_0}}.$$

With

$$K(t, \tau) = \begin{cases} |t - \tau|^{-\frac{(m+2)n-2}{(m+2)n+2}}, & \tau \geq 0, \\ 0, & \tau < 0, \end{cases}$$

it follows from Lemma 3.2 with  $q = q_0$  that (3.38) has been obtained.

Next we prove (3.38) for  $q = \infty$ . In this case, the kernel of  $A_j^{\alpha_1}$  can be written as

$$K_j^{\alpha_1}(t, x; \tau, y) = \int_{\mathbb{R}^n} \beta\left(\frac{|\xi|}{2^j}\right) e^{i((x-y) \cdot \xi + (\phi(t) - \phi(\tau))|\xi|)} a(t, \tau, \xi) \frac{d\xi}{|\xi|^{\alpha_1}}.$$

We now assert

$$\sup_{t, x} \int_{\mathbb{R}_+^{n+1}} |K_j^{\alpha_1}(t, x; \tau, y)|^{q_0} d\tau dy < \infty. \quad (3.40)$$

Obviously, if (3.40) is true, then a direct application of Hölder's inequality yields (3.38) for  $q = \infty$ .

Next we turn to the proof of (3.40). By [27, Lemma 7.2.4], we have

$$\begin{aligned} & \left| K_j^{\alpha_1}(t, x; \tau, y) \right| \\ & \leq C_{N, n, \alpha_1} \lambda^{\frac{n+1}{2} - \bar{\alpha}_1} (|\phi(t) - \phi(\tau)| + \lambda^{-1})^{-\frac{n-1}{2}} (1 + \lambda||x - y| - |\phi(t) - \phi(\tau)||)^{-N}, \end{aligned} \quad (3.41)$$

where  $\lambda = 2^j$ ,  $N = 0, 1, 2, \dots$ , and

$$\bar{\alpha}_1 = \frac{2}{m+2} + \alpha_1 = \frac{2}{m+2} + \frac{n}{2} - \frac{1}{m+2} = \frac{n}{2} + \frac{1}{m+2}.$$

It suffices to prove (3.40) in case  $x = 0$ . In fact, a direct computation yields

$$\begin{aligned}
& \int_{\mathbb{R}^{n+1}} |K_j^{\alpha_1}(t, 0; \tau, y)|^{q_0} d\tau dy \\
& \leq \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} \lambda^{(\frac{n+1}{2} - \alpha_1) \cdot q_0} (|\phi(t) - \phi(\tau)| + \lambda^{-1})^{-\frac{n-1}{2} \cdot q_0} (1 + \lambda||y| - |\phi(t) - \phi(\tau)||)^{-N} ds dy \\
& \leq C \int_{-\infty}^{\infty} \lambda^{\frac{m}{2(m+2)} \cdot q_0} (|\phi(t) - \phi(\tau)| + \lambda^{-1})^{-\frac{n-1}{2} \cdot q_0} \lambda^{-1} (|\phi(t) - \phi(\tau)| + \lambda^{-1})^{n-1} d\tau \\
& \leq C \int_{-\infty}^{\infty} \lambda^{\frac{m(m+2)n+2m}{(m+2)((m+2)n-2)} - 1} \left( |t - \tau| + \lambda^{-\frac{2}{m+2}} \right)^{-\frac{2(n-1)(m+2)}{(m+2)n-2}} d\tau \\
& \leq C.
\end{aligned}$$

Thus, by interpolation, (3.38) and then further (3.33) are shown.  $\square$

Relying on Lemmas 3.3 and 3.4, we have:

**Lemma 3.5.** *Let  $w$  solve (3.3). Then*

$$\|w\|_{L^q(\mathbb{R}_+^{n+1})} + \left\| |D_x|^{\gamma - \frac{1}{m+2}} w \right\|_{L^{q_0}(\mathbb{R}_+^{n+1})} \leq C \left\| |D_x|^{\gamma - \frac{1}{m+2}} F \right\|_{L^{p_0}(\mathbb{R}_+^{n+1})}, \quad (3.42)$$

where  $\gamma = \frac{n}{2} - \frac{(m+2)n+2}{q(m+2)}$ ,  $q_0 \leq q < \infty$ , and the constant  $C$  only depends on  $m, n$ , and  $q$ .

*Proof.* Note that

$$(\partial_t^2 - t^m \Delta) |D_x|^{\gamma - \frac{1}{m+2}} w = |D_x|^{\gamma - \frac{1}{m+2}} F.$$

Then applying Lemma 3.4 with  $q = q_0$  yields

$$\left\| |D_x|^{\gamma - \frac{1}{m+2}} w \right\|_{L^{q_0}} \leq C \left\| |D_x|^{\gamma - \frac{1}{m+2}} F \right\|_{L^{p_0}}.$$

Together with Lemma 3.3, this gives (3.42).  $\square$

## 4 Proof of Theorem 1.2

Based on the results of Section 3, here we shall prove Theorem 1.2. To establish the existence of a global solution of (1.1), we shall use the iteration scheme

$$\begin{cases} \partial_t^2 u_k - t^m \Delta u_k = |u_{k-1}|^p, \\ u_k(0, \cdot) = u_0(x), \quad \partial_t u_k(0, \cdot) = u_1(x), \end{cases} \quad (4.1)$$

where  $u_{-1} \equiv 0$ .

*Proof of Theorem 1.2.* We divide the proof into two parts.

**Part 1.**  $p_{\text{conf}}(m, n) \leq p \leq \frac{(m+2)(n-2)+6}{(m+2)(n-2)-2}$ .

We will show that there is a solution  $u \in L^r(\mathbb{R}_+^{n+1})$  of (1.1) with  $r = (\frac{m+2}{2}n + 1) \frac{p-1}{2}$  such that  $u_k \rightarrow u$  and  $|u_k|^p \rightarrow |u|^p$  in  $\mathcal{D}'(\mathbb{R}_+^{n+1})$  as  $k \rightarrow \infty$ .

We have that  $\frac{1}{m+2} \leq \gamma = \frac{n}{2} - \frac{(m+2)n+2}{r(m+2)} \leq 1 + \frac{1}{m+2}$  (using  $r \geq q_0$ ). Set

$$M_k = \|u_k\|_{L^r(\mathbb{R}_+^{n+1})} + \left\| |D_x|^{\gamma - \frac{1}{m+2}} u_k \right\|_{L^{q_0}(\mathbb{R}_+^{n+1})}. \quad (4.2)$$

Suppose that we have already shown that, for  $l = 1, 2, \dots, k$ ,

$$M_l \leq 2M_0 \leq C\epsilon_0. \quad (4.3)$$

Then we prove that (4.3) also holds for  $l = k + 1$ . Applying Lemma 3.4 to the equation

$$(\partial_t^2 - t^m \Delta)(u_{k+1} - u_0) = F(u_k),$$

where  $F(u_k) = |u_k|^p$ , we arrive at

$$\begin{aligned} M_{k+1} &\leq C \left\| |D_x|^{\gamma - \frac{1}{m+2}} (F(u_k)) \right\|_{L^{p_0}(\mathbb{R}_+^{n+1})} + M_0 \\ &\leq C \|F'(u_k)\|_{L^{\frac{(m+2)n+2}{4}}(\mathbb{R}_+^{n+1})} \left\| |D_x|^{\gamma - \frac{1}{m+2}} u_k \right\|_{L^{q_0}(\mathbb{R}_+^{n+1})} + M_0 \\ &\leq C \|F'(u_k)\|_{L^{\frac{(m+2)n+2}{4}}(\mathbb{R}_+^{n+1})} M_k + M_0. \end{aligned} \quad (4.4)$$

We mention that in this computation the following Leibniz's rule for fractional derivatives has been used (see [2, 3] for details):

$$\left\| |D_x|^{\gamma - \frac{1}{m+2}} F(u)(s, \cdot) \right\|_{L^{p_1}(\mathbb{R}^n)} \leq \|F'(u)(s, \cdot)\|_{L^{p_2}(\mathbb{R}^n)} \left\| |D_x|^{\gamma - \frac{1}{m+2}} u(s, \cdot) \right\|_{L^{p_3}(\mathbb{R}^n)}, \quad (4.5)$$

where  $\frac{1}{p_1} = \frac{1}{p_2} + \frac{1}{p_3}$  with  $p_i \geq 1$  ( $1 \leq i \leq 3$ ) and  $0 \leq \gamma - \frac{1}{m+2} \leq 1$ . Moreover, it follows from Hölder's inequality that

$$\|F'(u_k)\|_{L^{\frac{(m+2)n+2}{4}}(\mathbb{R}_+^{n+1})} \leq C \|u_k\|_{L^r(\mathbb{R}_+^{n+1})}^{p-1} \leq C M_k^{p-1} \leq C(2M_0)^{p-1}. \quad (4.6)$$

Thus, if  $M_0 \leq C\epsilon_0$  and  $\epsilon_0$  is so small that

$$C(2M_0)^{p-1} \leq \tilde{C}\epsilon_0^{p-1} \leq \frac{1}{2},$$

then we have

$$M_{k+1} \leq \frac{1}{2} M_k + M_0 \leq 2M_0.$$

Next we estimate  $M_0$ . By Lemma 3.3, we have that

$$M_0 \leq C \left( \|f\|_{\dot{H}^s(\mathbb{R}^n)} + \|g\|_{\dot{H}^{s - \frac{2}{m+2}}(\mathbb{R}^n)} \right) \leq C\epsilon_0, \quad (4.7)$$

where  $s = \frac{n}{2} - \frac{(m+2)n+2}{(m+2)r}$  and  $q_0 \leq r < \infty$ . Therefore, we have obtained the uniform boundedness of  $\{M_k\}$ .

Next we show that the sequence  $\{u_k\}$  is convergent under the weaker norm  $\|\cdot\|_{L^{q_0}(\mathbb{R}_+^{n+1})}$ . Set  $N_k = \|u_k - u_{k-1}\|_{L^{q_0}(\mathbb{R}_+^{n+1})}$ . Then

$$\begin{aligned} N_{k+1} &= \|u_{k+1} - u_k\|_{L^{q_0}(\mathbb{R}_+^{n+1})} \leq \|F(u_k) - F(u_{k-1})\|_{L^{p_0}(\mathbb{R}_+^{n+1})} \\ &\leq (\|u_k\|_{L^r(\mathbb{R}_+^{n+1})} + \|u_{k-1}\|_{L^r(\mathbb{R}_+^{n+1})})^{p-1} \|u_k - u_{k-1}\|_{L^{q_0}(\mathbb{R}_+^{n+1})} \\ &\leq (M_k + M_{k-1})^{p-1} \|u_k - u_{k-1}\|_{L^{q_0}(\mathbb{R}_+^{n+1})} \leq C\epsilon_0^{p-1} \|u_k - u_{k-1}\|_{L^{q_0}(\mathbb{R}_+^{n+1})} \\ &\leq \frac{1}{2} \|u_k - u_{k-1}\|_{L^{q_0}(\mathbb{R}_+^{n+1})} = \frac{1}{2} N_k \end{aligned} \quad (4.8)$$

Therefore,  $u_k \rightarrow u$  in  $L^{q_0}(\mathbb{R}_+^{n+1})$  and hence in  $\mathcal{D}'(\mathbb{R}_+^{n+1})$ . This yields that there exists a subsequence, which is still denoted by  $\{u_k\}$ , such that  $u_k \rightarrow u$  a.e. In view of  $\|u_k\|_{L^r(\mathbb{R}_+^{n+1})} \leq 2M_0$ , it follows from Fatou's lemma that

$$\|u\|_{L^r(\mathbb{R}_+^{n+1})} \leq \liminf_{k \rightarrow \infty} \|u_k\|_{L^r(\mathbb{R}_+^{n+1})} \leq 2M_0 \leq C\epsilon_0 < \infty.$$

It remains to prove that  $F(u_k) \rightarrow F(u)$  in  $\mathcal{D}'(\mathbb{R}_+^{n+1})$  in order to show that  $u$  is a solution of (1.1). In fact, for any fixed compact set  $K \Subset \mathbb{R}_+^{n+1}$ , one has

$$\begin{aligned} \|F(u_k) - F(u)\|_{L^1(K)} &\leq C_K \|F(u_k) - F(u)\|_{L^{p_0}(K)} \\ &\leq C_K (\|u_k\|_{L^r(\mathbb{R}_+^{n+1})} + \|u\|_{L^r(K)})^{p-1} \|u_k - u\|_{L^{q_0}(K)} \\ &\leq \tilde{C}_K \epsilon_0^{p-1} \|u_k - u\|_{L^{q_0}(K)} \rightarrow 0 \quad \text{as } k \rightarrow \infty. \end{aligned} \quad (4.9)$$

Thus  $|u_k|^p \rightarrow |u|^p$  in  $L_{loc}^1(\mathbb{R}_+^{n+1})$  and hence in  $\mathcal{D}'(\mathbb{R}_+^{n+1})$ .

The proof of Part 1 is complete.

**Part 2.**  $p \geq \frac{(m+2)(n-2)+6}{(m+2)(n-2)-2}$ ,  $p$  is an integer, and  $|u^p|$  in (1.1) is replaced with  $\pm u^p$ .

We will show that there is a solution  $u \in L^r(\mathbb{R}_+^{n+1})$  of (1.1) with  $r = \left(\frac{m+2}{2}n + 1\right) \frac{p-1}{2}$  such that  $u_k \rightarrow u$  and  $u_k^p \rightarrow u^p$  in  $\mathcal{D}'(\mathbb{R}_+^{n+1})$  as  $k \rightarrow \infty$ .

We have that  $\gamma = \frac{n}{2} - \frac{(m+2)n+2}{(m+2)r} > 1 + \frac{1}{m+2}$ . Let

$$M_k = \sup_{q_0 \leq q \leq r} \left\| |D_x|^{\frac{(m+2)n+2}{q(m+2)} - \frac{2}{m+2} \cdot \frac{2}{p-1}} u_k \right\|_{L^q(\mathbb{R}_+^{n+1})}. \quad (4.10)$$

Applying Lemma 3.4 to the equation

$$(\partial_t^2 - t^m \Delta)(u_{k+1} - u_0) = |u_k|^p$$

yields

$$M_{k+1} \leq M_0 + C_p \left\| |D_x|^{\frac{n}{2} - \frac{1}{m+2} - \frac{2}{m+2} \cdot \frac{2}{p-1}} |u_k|^p \right\|_{L^{p_0}(\mathbb{R}_+^{n+1})}. \quad (4.11)$$

To treat the second summand on the right-hand side of (4.11), we need the following variant of (4.5) (see [16] for details):

$$\| |D_x|^\sigma (fg) \|_{L^p} \leq C \| |D_x|^\sigma f \|_{L^{r_1}} \|g\|_{L^{r_2}} + C \|f\|_{L^{s_1}} \| |D_x|^\sigma g \|_{L^{s_2}}, \quad (4.12)$$

where  $0 \leq \sigma \leq 1$ ,  $1 < r_j, s_j < \infty$ , and  $\frac{1}{p} = \frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{s_1} + \frac{1}{s_2}$ .

By (4.12) together with the fact that, for a given multi-index  $\alpha$  and  $1 < p < \infty$ ,

$$\|D_x^\alpha f\|_{L^p} \leq C_{p,\alpha} \left\| |D_x|^{|\alpha|} f \right\|_{L^p},$$

we arrive at

$$\left\| |D_x|^{\frac{n}{2} - \frac{1}{m+2} - \frac{2}{m+2} \frac{2}{p-1}} (|u_k|^p) \right\|_{L^{p_0}(\mathbb{R}_+^{n+1})} \leq C \prod_{j=1}^p \| |D_x|^{\alpha_j} u_k \|_{L^{q_j}(\mathbb{R}_+^{n+1})},$$

where  $0 \leq \alpha_j \leq \frac{n}{2} - \frac{1}{m+2} - \frac{2}{m+2} \frac{2}{p-1}$  and

$$\sum_{j=1}^p \alpha_j = \frac{n}{2} - \frac{1}{m+2} - \frac{2}{m+2} \frac{2}{p-1}. \quad (4.13)$$

Let  $q_0 \leq q_j < \infty$  satisfy

$$\sum_{j=1}^p \frac{1}{q_j} = \frac{1}{p_0}, \quad (4.14)$$

where  $q_j$  is determined by

$$\frac{(m+2)n+2}{q_j(m+2)} - \frac{2}{m+2} \frac{2}{p-1} = \alpha_j.$$

From this, we have

$$q_0 \leq q_j \leq \frac{(m+2)n+2}{4} (p-1)$$

and

$$\begin{aligned} \sum_{j=1}^p \frac{1}{q_j} &= \frac{m+2}{(m+2)n+2} \sum_{j=1}^p \left( \alpha_j + \frac{2}{m+2} \cdot \frac{2}{p-1} \right) \\ &= \frac{m+2}{(m+2)n+2} \left( \frac{n}{2} - \frac{1}{m+2} - \frac{2}{m+2} \frac{2}{p-1} + \frac{2p}{m+2} \frac{2}{p-1} \right) \\ &= \frac{1}{p_0}. \end{aligned} \quad (4.15)$$

Thus one has from (4.11) that

$$M_{k+1} \leq M_0 + C_p M_k^p.$$

Suppose that  $M_k \leq 2M_0 \leq C\epsilon_0$  holds. Then

$$M_{k+1} \leq M_0 + C_p (2M_0)^{p-1} M_k \leq M_0 + \tilde{C}_p \epsilon_0^{p-1} M_k.$$

If  $\epsilon_0 > 0$  is so small that  $\tilde{C}_p \epsilon_0^{p-1} \leq 1/2$ , then

$$M_{k+1} \leq M_0 + \frac{1}{2} M_k \leq M_0 + \frac{1}{2} \cdot 2M_0 = 2M_0.$$

Thus, we have obtain the uniform boundedness of the  $M_k$  provided that  $M_0 \leq C\epsilon_0$ .

Furthermore, we then have that, if  $N_k$  is defined as in (4.8),

$$\begin{aligned}
N_{k+1} &= \|u_{k+1} - u_k\|_{L^{q_0}(\mathbb{R}_+^{n+1})} \\
&\leq \| |u_k|^p - |u_{k-1}|^p \|_{L^{p_0}(\mathbb{R}_+^{n+1})} \\
&\leq \left( \|u_k\|_{L^r(\mathbb{R}_+^{n+1})} + \|u_{k-1}\|_{L^r(\mathbb{R}_+^{n+1})} \right)^{p-1} \|u_k - u_{k-1}\|_{L^{q_0}(\mathbb{R}_+^{n+1})} \\
&\leq \left( \sup_{q_0 \leq q \leq r} \left\| |D_x|^{\frac{(m+2)n+2}{q(m+2)} - \frac{2}{m+2} \cdot \frac{2}{p-1}} u_k \right\|_{L^q(\mathbb{R}_+^{n+1})} \right. \\
&\quad \left. + \sup_{q_0 \leq q \leq r} \left\| |D_x|^{\frac{(m+2)n+2}{q(m+2)} - \frac{2}{m+2} \cdot \frac{2}{p-1}} u_{k-1} \right\|_{L^q(\mathbb{R}_+^{n+1})} \right)^{p-1} \|u_k - u_{k-1}\|_{L^{q_0}(\mathbb{R}_+^{n+1})} \\
&\leq (M_k + M_{k-1})^{p-1} \|u_k - u_{k-1}\|_{L^{q_0}(\mathbb{R}_+^{n+1})} \\
&\leq C \epsilon_0^{p-1} \|u_k - u_{k-1}\|_{L^{q_0}(\mathbb{R}_+^{n+1})} \\
&\leq \frac{1}{2} \|u_k - u_{k-1}\|_{L^{q_0}(\mathbb{R}_+^{n+1})} = \frac{1}{2} N_k.
\end{aligned}$$

Thus,  $u_k \rightarrow u$  in  $L^{q_0}(\mathbb{R}_+^{n+1})$  as  $k \rightarrow \infty$ . From here we can finish the proof of Part 2 as in Part 1.

Part 1 and Part 2 jointly constitute the proof of Theorem 1.2.  $\square$

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